Set notation

- A set is a collection of objects called elements.
  If a is an element of the set S, we write
  $$a \in S,$$
  and say that a belongs to S. If b does not belong to S, we write $b \notin S$.

- Sets can be specified in two ways:
  (a) Listing its elements
  $$A = \{x_1, x_2, \dots, x_n\} \qquad \text{finite}$$

  (b) Stating the property that every element must satisfy:

$B = \{x : 1 < x < 2\}$ infinite sets

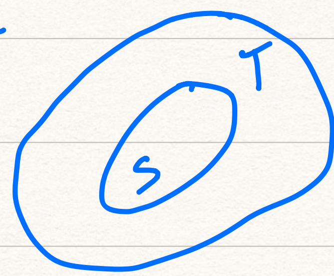
- The symbol $\emptyset$ is used to denote the empty set.

$$\{x : x^2 + 1 = 0\} = \emptyset$$

$$x^2 + 1 = 0 \Rightarrow x^2 = -1$$

- Let S and T be two sets. If every element of S also belongs to T, we say that S is a subset of T and write

$$S \subseteq T.$$



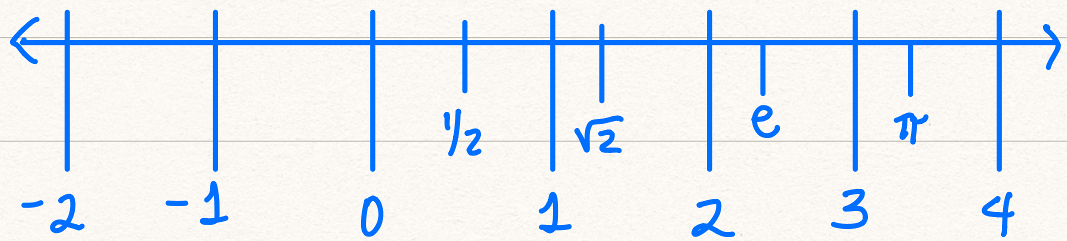
Every $a \in S$ also $a \in T$

- Two sets A and B are equal, denoted by $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

$$A = B \Rightarrow \begin{cases} A \subseteq B \\ \text{and} \\ B \subseteq A \end{cases}$$

The set of real numbers $\mathbb{R}$.

Usually represented as a straight, solid line that extends indefinitely in both directions.



Arithmetic

For every $a, b, c \in \mathbb{R}$, the following properties hold:		
	Addition:	Multiplication:
Commutativity:	$a + b = b + a$	$a\,b = b\,a$
Associativity:	$a + (b + c) = (a + b) + c$ $= a+b+c$	$a\,(b\,c) = (a\,b)\,c = abc$
Identity:	There is a real number 0, such that $a + 0 = a$.	There is a real number 1, such that $1\,a = a$.
Inverse:	For each a there exists $u \in \mathbb{R}$, such that $a + u = 0$.	For each $a \neq 0$ there exists $v \in \mathbb{R}$, such that $a\,v = 1$.
Distributivity:	$a\,(b + c) = a\,b + a\,c.$	

Subtraction and division are defined as

$$a - b = a + \underbrace{(-b)}_{\text{additive inverse of } b} \quad \text{and} \quad \frac{a}{b} = a\underbrace{b^{-1}}_{\text{multiplicative inverse}} \quad (b \neq 0)$$

where $-b$ and b^{-1} are the additive and multiplicative inverse of b, respectively.

- Division by 0 is not defined. The expression $\frac{2}{0}$ makes no sense.

- ∞ is not a real number and it is not true that $\frac{2}{0} = \infty$.

$\boxed{\text{Theorem}}$

The additive and multiplicative identities are unique.

Proof: Suppose there is some $u \in \mathbb{R}$ such that $a+u=a$ for all $a \in \mathbb{R}$

We will prove that $u=0$.

$$a+u=a \quad \forall a \in \mathbb{R}$$

$a=0 \qquad u=0+u=0$

Suppose that there is some $v \in \mathbb{R}$ such that $va=a^* \quad \forall a \in \mathbb{R}$

If I fix $a=1$

$$v = v \cdot 1 \overset{*}{=} 1 \Rightarrow v=1$$

because 1 is an identity

$\boxed{\text{Theorem}}$

Let a be any real number. Then a has a unique additive inverse. If $a \neq 0$, it has a unique multiplicative inverse.

<u>Proof:</u> Fix $a \in \mathbb{R}$. $(-a)$ $a+(-a)=0$

Suppose that u is also an additive inverse of a. We will prove that $u=-a$

$$u = u+0 = u+[a+(-a)] \overset{\text{Associativity}}{=} (u+a)+(-a)$$

$$(u+a=0) \quad = 0+(-a) = -a \Rightarrow \boxed{u=-a}$$

Suppose that v is also a mult. inverse of a. (a^{-1} and $a \cdot a^{-1} = 1$)

$$v = 1v = (a^{-1} \cdot a)v \overset{\text{Associativity}}{=} a^{-1}(av) \underset{av=1}{=} a^{-1} \cdot 1 = a^{-1}$$

$$\boxed{v = a^{-1}}$$

Theorem

For any $x \in \mathbb{R}$, if $\underset{(\text{hyp})}{a+x=b+x}$, then $a=b$.

Proof:

$$
\begin{aligned}
a &= a+0 \\
&= a+(x-x) \\
&= (a+x)-x \\
(\text{hyp.}) \quad &= (b+x)-x \\
&= b+(x-x) \\
&= b+0 \\
&= b
\end{aligned}
$$

$\boxed{\text{Theorem}}$

1. For any non-zero $x \in \mathbb{R}$, if $ax = bx$, then $a = b$.
2. $0x = 0$ for all $x \in \mathbb{R}$
3. $1 \neq 0$.
4. $(-1)x = -x$ for all $x \in \mathbb{R}$.
5. $-(-x) = x$ for all $x \in \mathbb{R}$.
6. If $xy = 0$, then either $x = 0$ or $y = 0$.
7. For all $x, y \in \mathbb{R}$, $x(-y) = -(xy)$.
8. For all $x, y \in \mathbb{R}$, $(-x)(-y) = xy$.
9. If $x \neq 0$, then $x^{-1} \neq 0$ and $(x^{-1})^{-1} = x$.
10. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = x^{-1}y^{-1}$.
11. For any non-zero $x \in \mathbb{R}$, $(-x)^{-1} = -x^{-1}$